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SINGULARLY PERTURBED TWO- POINT BOUNDARY VALUE PROBLEM BY APPLYING HYPERBOLIC DECENT DYNAMIC METHOD

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Abstract: **We are proposed to introduce and describe a exponential decent factor (shift operator) in this research paper to resolve a specifically defined two point boundary value problem with end limit layer in the vicinity [0,1]. Here, we've applied the finite difference related method with suitably selected hyperbolic fitting factor for the convergence of the numerically computed solution in accordance with the analytical solution. Due to the selected fitting factor method the numerical solution will give us a stable, consistent and convergent approximate solution in the regular region and more particularly in the boundary layer region also. We are also observed that the error bounds in the proposed numerical method were considerably lower.**

Also we have investigated the aptness of the defined mathematical model, difference between computed and analytical solution. we have applied the suitable numerical algorithm called as a technique on defined mathematical model with a right/left end boundary layer that was tested and incredibly good arrangement with the exact solution which is available in the literature. Meanwhile the numerically computed results were compared with the analytical solutions for exactness and to evaluate the error bound throughout the domain. For the assembly of the proposed method the differential equation is transformed in to a difference equation with suitable multiplication factor.

Computational results are closely associated with the analytical solutions available in the literature. By enlarge this method is subsidiary method to solve two point boundary value problems with reasonable accuracy.

Keywords: **Singular Perturbation problem; Ordinary Differential Equation; Boundary Layer; Two-Point Boundary Value Problem; Exponential fitting factor, Convergence analysis , Error Bound.**

I. INTRODUCTION

In the mathematical model of a target system, as in control systems, the presence of parasitic short-time hyperbolic fitting factor consists some parameters such as moments of inertia, resistances, inductances and capacitances increases the order and rigidity of these systems. The suppression of these small constants results in the reduction of the order of the defined system. Such systems are called singular perturbation systems and when these systems take into account both the past history and the current state of the physical system, they are called singularly perturbed delay differential equations. Differential delay equations have arisen in the past in neuro-biology problems and in the mathematical formulation of various practical phenomena in bio-sciences. The study of differential difference equations, with the presence of displacement terms, which induce large amplitudes and exhibit oscillations, resonance, inflection point behaviour. Means due to small parameter the solution nature is stable at some region and Unstable or oscillatory in the boundary layer region. Such kind of qualitative analysis and error bounds are studied. .

In this research work, the numerical process is studied with a cumulative hyperbolikc exponential adjustment factor for the solution o[redacted]perturbed differential equation latter on conv[redacted]nce equation with a hyperbolic displacement called the one-sided delay differential equation, with layer behaviour. A layer is one that is a narrow region where the nature of the solution changes as rapidly as a two sided exponential curve. Initially we have considered that the negative displacement in the derivative term is approximated using the Taylor series for a variable function as long as the displacement is O (ε). Subsequently, the differential delay equation is replaced by an asymptotically comparable first-order neutral delay differential equation. An Hyperbolic integration factor is introduced into the first order delay equation. Then, the Weddels rule has been used together with linear interpolation to obtain a three-term recurrence relation. The resulting tri-diagonal system is solved by using the efficient Thomas algorithm. The selected technique is implemented in some selected problems, for different values of the delay parameter δ and the perturbation parameter ε. The maximum absolute errors are tabulated and compared to validate the technique. The convergence of the proposed method has also been discussed. About the oscillatory nature of the problem we have taken literature on boundary and interior layers was carried out by Lange and Miura [4]. Extensive numerical work has been initiated by N. Srinivasacharyulu [12], P.B. Patil [4], and J. B. M.C . Cartin [9], Bender [2] carried out so many related proposed a numerical methods comprising an finite difference approximation on an selected grid with an curved type monitor function, to approximate the solutions of singularly perturbed differential-difference equations with small delay and shift terms. Geng and etal. [6] presented an improved kernel method to obtain an accurate approximation of solutions for singularly perturbed differential-difference equations with small delay. With this motivation, in this paper we employed a numerical technique with the exponential integrating factor for the solution of singularly perturbed delay differential equations, with layer behaviour. A numerical scheme for the solution of a singularly perturbed delay differential equation with the left-end boundary layer and the right-end boundary layer is described .convergence of the proposed method is analyzed. To demonstrate the efficiency of the method, numerical experiments are carried out for several test problems and the results are tabulated and compared in the discussion. Finally, the advantages and pitfalls and conclusion are given in the last section.

II. MATHEMATICAL FORMULATION & NUMERICAL ALGORITHM :

Let us define the delay proposed differential equation with layer behaviour

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x) \qquad (1)$$

in the interval (0, 1) with the prescribed conditions

$$y(x) = \varphi(x) \text{ for } -\delta \le x \le 0 \qquad (2)$$

$$y(1) = \beta, \qquad (3)$$

where ε is a small positive parameter $0 < \varepsilon \ll 1$, a(x), b(x), f (x) and $\varphi(x)$ are sufficiently smooth functions and β is a positive constant. Furthermore, $\delta = o(\varepsilon)$, where δ is a delay parameter. When δ is zero, equation (1) reduces to a singular perturbation problem, where as small ε reveals boundary layer and turning points depends upon the coefficient of convection term. The layer behavior of the problem under consideration is maintained only for δ when f becomes nullity means sufficiently small, i.e., δ is of (ε). When the delay parameter exceeds the perturbation parameter, i.e., δ is of $O(\varepsilon)$, then layer behavior of the solution is no longer maintained, rather the solution exhibits an oscillatory behavior or diminished behavior Lange and Miura [7].

2(a) Negative end boundary layer problems:

The solution of the problem (1)-(3) exhibits a boundary layer at one end of the interval depending on the sign of a(x). We assume that $a(x) \ge M > 0$ through- out the interval [0, 1], for some positive constant M. This assumption implies that the boundary layer will be in the vicinity of x=0.

By using a Taylor series expansion of the retarded term $y'(x-\varepsilon)$ in the neighbour- hood of the point x, we have

$$y'(x-\varepsilon) \approx y'(x) - \varepsilon y''(x) \qquad (4)$$

as a result, equation (1) is replaced by an asymptotically equivalent first order delay neutral type differential equation

$$y'(x) + b(x)y(x) = f(x) + y'(x-\varepsilon) - a(x)y'(x-\delta) \qquad (5)$$

with $y(0) = \varphi(0)$; $y(1) = \beta$. Since $0 < \delta \ll 1$, the transition from (1) to (5) is accepted. This replacement is significant from the computational point of view. For more details on the validity of this transition, one can refer to El'sgolts and Norkin [2]. Thus, the solution of equation (5) provides a reasonable approximation to the actual solution.

of equation (1). Here, for consolidation of our ideas, we assume a(x) and b(x) to be constants.

By applying an integrating factor e^{bx} to equation (5)

$$\frac{d}{dx}(e^{bx}.y(x) = e^{bx}.\{f(x) + y'(x-\varepsilon) - ay'(x-\delta)\}dx \qquad (6)$$

Discrediting the interval [0,1] into N equal subintervals of mesh size h = 1/N, let $0 = x_0, x_1, \ldots, x_N = 1$ be the mesh

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points. Then we have $x_i = ih$, for $i = 0, 1, \ldots, N$. Integrating (6) with respect to the independent variable x from x_i to x_{i+1} after multiplying both sides with a permissible hyperbolic natured fitting factor (b is real) i.e.,

$$\int_{x_i}^{x_{i+1}} a \frac{dy}{dx} \exp^{(bx)} dx =$$

$$a\int_{x_i}^{x_{i+1}} \exp^{(bx)}.\{f(x) + y'(x-\varepsilon) - ay'(x-\delta)\}dx \qquad (7)$$

$ae^{bx_{i+1}y_{i+1}} - e^{bx_i y_i} = a \int_{x_i}^{x_{i+1}} e^{bx}.\{f(x) + y'(x-\varepsilon) -$ ay′x−δdx+ebxi+1 yi+1−ε-ebxiyxi−ε−aebxi+1 yxi+1−δ+a ebxi yxi−δ

(8)

Here a is a parameter. Term by term integration is admitted because e^{bx},f(x) and the derivatives are continuous. Perfoming the integration we have .

using Weddles rule the number of sub intervals should be taken as multiple of 3n i.e always select the interval of the type (x_0, x_{3n}) , n=2,,4, 6…...2m (m=1,2,3....)

$ae^{bx_{i+1}} y_{i+1} - e^{bx_i} y_i = 3h/10 \{ e^{bx_i}.\{f(x_i) +$ y′xi−ε−ay′xi−δ+5ebxi−1.fxi−1+y′xi−1−ε−ay′xi−1−δ +2ebxi+1.fxi+1+y′xi+1−ε−ay′xi+1−δ

(9)

further, since $0 < \varepsilon \ll 1$ and $\delta = o(\varepsilon)$, to tackle the terms containing delay, the virtue of Taylor series approximations ,approximating $y'(x)$ by linear interpolation, we get

$$y(x_i - \delta) = y(x_i) - \delta y'(x_i) = (1-\frac{\delta}{h})\, y(x_i) + \frac{\delta}{h} y_{i-1} \qquad (10)$$

$$y(x_{i+1} - \delta) = y(x_{i+1}) - \delta y'(x_{i+1}) = (1-\frac{\delta}{h})y(x)_{i+1} + \frac{\delta}{h} y(x)_i \quad - \qquad (11)$$

$$y(x_i - \varepsilon) = y(x_i) - \varepsilon\, y'(x_i) = (1-\frac{\varepsilon}{h})y_i + \frac{\varepsilon}{h} y_{i-1} \qquad (12)$$

$$y(x_{i+1} - \varepsilon) = y(x_{i+1}) - \varepsilon y'(x_{i+1}) = (1-\frac{\varepsilon}{h})y_{i+1} + \frac{\varepsilon}{h} y_i \qquad (13)$$

using equations (10) - (13) in equation (9) and simplifying, we get

$$\text{Exp}(^{bh}) \left[\frac{\varepsilon}{h} + \frac{hb}{2}\left\{1 - \frac{\varepsilon}{h}\right\} + a\left[1 - \frac{\delta}{h}\right] + hab/2[1 - \frac{\delta}{h}\right] y_{i+1} - e^{bh}\left[1 + \frac{\varepsilon}{h} - 0.5b\varepsilon - \frac{a\delta}{2}\right] - (1 - \frac{\varepsilon}{h})(1+bh/2) + a(1-\frac{\delta}{h}) + 0.5hba]y_i$$

$$+(\frac{\varepsilon}{h} + 0.5b\,\varepsilon - \frac{a\delta}{h} + \frac{ab\delta}{2})y_{i-1} = 0.5h[\, f_{i+1} + \exp(bh)]$$

(14)

The resulting three-term recurrence relation of (14) is of the form

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i;\ i = 1, 2, \ldots, N-1 \qquad (15)$$

Subject to the boundary conditions

Where

$$E_i = \frac{\varepsilon}{h} + 0.5b\,\varepsilon - \frac{a\delta}{h} + \frac{ab\delta}{2}. \qquad (16)$$

$$F_i = -e^{bh}\left[1 + \frac{\varepsilon}{h} - 0.5b\varepsilon - \frac{a\delta}{2}\right] - (1 - \frac{\varepsilon}{h})(1+bh/2) + a(1-\frac{\delta}{h}) + 0.5hba \qquad (17)$$

$$G_i = \text{Exp}(^{bh}) \left[\frac{\varepsilon}{h} + \frac{hb}{2}\left\{1 - \frac{\varepsilon}{h}\right\} + a\left[1 - \frac{\delta}{h}\right] + hab/2[1 - \frac{\delta}{h}\right]$$

$$H_i = 0.5h[\, f_{i+1} + exp(bh) \qquad \{ 18 \}$$

Unknowns y_0 , y_1 ,........ y_n which will be the sufficient to solve for these unknowns. The matrix problem associated here is a tridigonal algebriac system whose solution can easily be determined by an efficient algorithm is called Thomas algorithm. The idea of this algorithm is very simple. We shall briefly describe it in the following. In this algorithm we start with a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \ , \quad i = 0, 1, \ldots\ldots, n-1 \ , \qquad (19)$$

Where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ are to be determined. By using (19) in (15) , we see that the recurrence relations W_i and T_i for $i = 0, 1, \ldots\ldots, n-1$ are obtained as

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}}$$

(20)

and

$$T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}$$

(21)

To solve these recurrence relations W_i and T_i for $i = 0, 1, \ldots\ldots, n-1$, we need to know the initial cond[redacted]

$$W_0 = \frac{S_1}{S_0}, \quad (22)$$

$$T_0 = \frac{S_2}{S_0}$$

(23)

The numerically computed results are tabulated and compared with the exact solutions available in the literature.

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III. ERROR ANALYSIS

Let y and Y be the exact solution of (1) and the approximate solution of the same governing equation (1) respectively. For adequate upper value N, we can estimate the ε- Uniform error bound.

$\llbracket Supremum_{0<\varepsilon\leq 1}$ K $\|y-Y\| \leq CN^{-1}(lnN)^2$, $x \in$ R (24)

K is suitable positive integer.

In order to get the inequality let decompose the solution y(x) of governing equation (1) into regular and singular segments as :

y(x) = re(x) + sp(x)

for $0 \leq k \leq 3$ the regular component re(x) satisfies the inequality as

$|r^k(x)] \leq C[1+\varepsilon^{2-k}e(x,a), \forall\, x \in [0,1]$ (25)

Also the singularly oscillatory component sp(x) follows the inequality as

$|r^k(x)] \leq C\varepsilon^{-k}e(x,a)\ \ \forall\, x \in [0,1]$

Where e(x,a) is the hyperbolic component have two parts vig.,

e(x,a) = $e^{\frac{-a_0(x)}{\varepsilon}} + e^{\frac{-a_0(1-x)}{\varepsilon}}$ coined by Miller, J.J.H etal [10] { In Mahapatra}

subsequently decompose the associated Numerical solution Y(x) of the defined problem (1) into Regular ($R(\varepsilon)$ $and\ singlarly\ perturbed\ S(\varepsilon)$ components i.e

Y(x) = $R(\varepsilon,x) + S(\varepsilon,x)$. Here $R(\varepsilon,x)$ & $S(\varepsilon,x)$ are the approximate solutions of the defined problems as below

$L^N R(\varepsilon,x) = f(x),\ \ R(\varepsilon,0) = r(0),\ \ R(\varepsilon,1) = r[1]$

$L^N S(\varepsilon,x) = 0\ ,\ \ S(\varepsilon,0) = s(0),\ \ S(\varepsilon,1) = s[1]$

So that $\|y(x)-Y(x)\| \leq \|r(x)-R(\varepsilon,x)\| + \|s(x)-S(\varepsilon,x)\|$

Afterwards we have a necessity to calculate the erros occurred in the linear (regular) and singularly perturbed components respectively

First evaluate the error in the regular region component

The local Truncation error defined as

$L^N[R(\varepsilon,x)$ -r(x)] = $(L-L^N)r(x) = f(x) - L^N[r(x)]$ =

$\varepsilon\,[D^2-\Delta^2]r(x) + a(x)[D-D^0]r(x)$ (C) (23)

With the virtue of Taylor's series expansion by omitting the higher order terms after third order , one can obtain the expansions for

$y\,(x_i+h)\ and\ y\,(x_i-h)$ as

$y\,(x_i+h) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{3!}(y'''(\xi_1^{(i)})$

also

$y\,(x_i-h) = y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{3!}(y'''(\xi_2^{(i)})$

where $\xi_1^{(i)}$ & $\xi_2^{(i)} \in (x_{i-1}, x_{i+1})$

From the above two converging expansions in the given region R we have

$(\Delta^2 y)(x_i) = y''(x_i) - \frac{h}{6}[y'''(\xi_1^{(i)}$ - $y'''(\xi_2^{(i)})]$

Whence $\left\|\left(\Delta^2 - \frac{d^2}{dx^2}\right)y(x_i)\right\| \leq C\|y'''\|$ where $\|y'''\| = Sup_{x_i\in(x_0,x_N}|y'''(x_i)|$

Again by using the Taylor's expansion restricting the terms upto second order one can get

$$\left\|\left(D^0 - \frac{d}{dx}\right)y(x_i)\right\| \leq C\|y''\|$$

By using the bounds of $r^k(x)$, $s^k(x)$ and the validity of the assumption

$\varepsilon \leq CN^{-1}$ the equation (23) reduced as

$\|(L^N(R_\varepsilon - r(x_i)\| \ \leq CN^{-1}$ (26)

Also using the approximate discrete maximum principle one can reach the inequality

$\|(R_\varepsilon - r)\| \ \leq CN^{-1}$ (27)

Finally we required to calculate the error bound in the singular perturbed component. The local truncation error exists in the singular component is bounded in the regular way as derived for the regular analytical part and is

$\|(L^N(S_\varepsilon - s)(x_i)\| \ \leq CN^{-1}\varepsilon^{-2}$ (28)

Which completes the convergence analysis and upper error bound can be calculated. Using the inequality [28].

IV. ILLUSTRATED PROBLEMS

Example 1. Consider the singularly perturbed differential difference equation ex- hibiting left-end boundary layer [4]:

$\varepsilon y''(x) + y'(x) - y(x) = 0$; $0 < x < 1$, y(0)=1 , y(1) =1 (A)

The exact solution of (A) with the given boundary conditions is

$y(x) = \frac{(exp\,(r_2-1)exp(r_1x) - (1-exp\,(r_1))(exp\,r_2x)}{exp\,(r_2) - exp\,(r_1)}$

(B)

In equation (B) $r_1 = \frac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon}$ and $r_2 = \frac{-1-\sqrt{1+4\varepsilon}}{2\varepsilon}$

This selected boundary value problem has a boundary layer at left end at x=0 . It is clearly understood by drawing its patterns as well as the numerical results computed in the below table with Maimum point-wise error & the rate of convergence E_ε^n and r_ε^n respectively .

Table -1

ε	Number of Intervals N					
	16	32	64	128	256	512
$1e^{-2}$	$7.585e^{-3}$ 0.9765	$3.8543e^{-3}$ 1.01108	$1.9127e^{-3}$ 1.0461	$9.2599e^{-}$ 1.0183	4.5788 e^{-4} 1.0637	$2.1880e^{-4}$
$1e^{-4}$	$1.1138e^{-2}$ 0.982918	$5.6348e^{-3}$ 0.9994	$2.8188\ e^{-3}$ 1.01445	$1.3953e^{-}$ 1.0366	6.8264 e^{-4} 1.07413	$3.2408e^{-4}$
$1e^{-8}$	$1.11708e^{-2}$ 0.9786	$5.67709e^{-2}$ 0.989618	$2.8564e^{-2}$ 0.9959	1.42359 e^{-3} 0.9979	$7.1809e^{-}$ 0.9939	$3.64488e^{-4}$
E^N	$1.11729e^{-2}$	$5.679e^{-2}$	$2.8564e^{-2}$	$1.4326e^{-}$	7.1795 e^{-4}	0
r^N	0.9784	0.9898	0.9960	0.9979	0.9940	1

Example -2: Select the Non linear singularly perturbed problem

$\varepsilon y''(x) + 2y'(x) + \exp(y(x)) = 0$; $0 < x < 1$, $y(0)=0$, $y(1) = 0$

The uniformly convergent validity solution after linearization by using quasi linearization due to Bellman and Kalaba [1] is

$Y(x) = \ln\left(\left(\frac{2}{1+x}\right) - (\ln 2)\exp\left(\frac{-2x}{\varepsilon}\right)\right)$ which reveals a boundary layer of thickness O(ε) in the neighbourhood of x=0 . the numerically computed results are presented in Table-2.

Table -2

ε	Number of Intervals N					
	16	32	64	128	256	512
$1e^{-4}$	$1.9276e^{-2}$ 0.9265	$1.0135e^{-2}$ 0.9645	5.28487 e^{-3} 0.9825	$2.599e^{-3}$ 0.9937	$1.34788e^{-3}$ 1.0008	$6.7540e^{-4}$
$1e^{-4}$	$1.1138e^{-2}$ 0.982918	$5.6348e^{-3}$ 0.9994	2.8188 e^{-3} 1.01445	$1.3953e^{-3}$ 1.0366	$6.8264e^{-4}$ 1.07413	$3.2408e^{-4}$
$1e^{-8}$	$1.96708e^{-2}$ 0.9286	$1.03119e^{-2}$ 0.969618	$5.2843e^{-3}$ 0.9859	2.6755 e^{-3} 0.9929	$1.344e^{-3}$ 0.9939	$6.7547e^{-4}$
E^N	$1.9629e^{-2}$	$1.0311e^{-2}$	$5.28564e^{-}$	2.671795 e^{-3}	1.344	0
r^N	0.9284	0.9698	0.9820	0.9930	0.9939	1

V. DISCUSSIONS AND CONCLUSION

We have defined , described and demonstrated the methodology of the proposed method for solving singular boundary value problems. We have defined the Hyperbolic type fitting factor and its validity on the real line, constructued this method by using the principle of newton –quotes term by term integramtion approximation and finally designed a descrete three term recursive relation. Such three term recursive difference equaitons is solved using efficient Gauss elimination type Thomas algorithm. This method is simple, accurate and user friendly to implement on computer. It is a practical method and can be easily implemented on computer to solve such problems. We have implemented this method with examples – a linear singular perturbed boundary problem, and non –linear singularly perturbed boundary value problem. The obtained computational results are tabulated and the numerical results compared with the exact solution and done the error analyais. . The numerically computed results are in good agreement with the exact solutions avaialble in the literatue.. The convergence of the solution is uniform through out the specified region which can be observed in the numerically computed results.

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